SPACE- AND TIME-CORRELATIONS IN THE SUPERNOVA DRIVEN INTERSTELLAR MEDIUM

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Abstract

We apply correlation analysis to random velocity, density and magnetic fields in numerical simulations of the supernova-driven interstellar medium (ISM). We solve the thermo-magneto-hydrodynamic (MHD) equations in a shearing, Cartesian box representing a local region of the ISM, subject to thermal and kinetic energy injection by supernova explosions, and parametrized optically-thin radiative cooling. We consider the cold, warm and hot phases of the ISM separately; the analysis mostly considers the warm gas, which occupies the bulk of the simulation volume. Various physical variables have different correlation lengths in the warm phase: 40pc, 50pc, and 60pc for magnetic field, density, and velocity, respectively, in the midplane. The correlation time of the random velocity field is comparable to the eddy turnover time, about 10^7 yr, although it may be shorter in regions with higher star formation rate. The random magnetic field is anisotropic, with the standard deviations of its components b_x/b_y/b_z having the approximate ratios 0.5/0.6/0.6 in the midplane. The anisotropy is attributed to the global velocity shear from galactic differential rotation, and locally inhomogeneous outflow to the galactic halo. The correlation length of Faraday depth along the z-axis, 120pc, is greater than for electron density, 60–90pc, and vertical magnetic field, 60pc. Such comparisons may be sensitive to the orientation of the line of sight. Uncertainties of the structure functions of the synchrotron intensity rapidly increase with the scale. This feature is hidden in power spectrum analysis, which can undermine the usefulness of power spectra for detailed studies of interstellar turbulence.

Keywords: galaxies: ISM – ISM: kinematics and dynamics – ISM: magnetic fields – turbulence – methods: statistical

1. INTRODUCTION

The interstellar medium (ISM) of a spiral galaxy is a complex, multiphase, random system, driven by the input of thermal and kinetic energy from supernova (SN) explosions and stellar winds (e.g., Mac Low & Klessen 2004; Elmegreen & Scalo 2004; Scalo & Elmegreen 2004; Mac Low et al. 2005; de Avillez & Breitschwerdt 2005; Federrath et al. 2010; Hill et al. 2012). Its statistical analysis, including that of interstellar turbulence, is complicated by the multi-phase structure, where the diversity of physical processes predominant in different phases causes strong inhomogeneity. Furthermore, interstellar turbulence is transonic or supersonic (Bykov & Top-tygin 1987; Vázquez-Semadeni 2015). The compressibility and abundance of random shock waves lead to spatial and temporal intermittency of the random velocity and magnetic fields and of the density fluctuations. Dynamo action adds further complexity by producing intermittent random magnetic fields (Wilkin et al. 2007).

Observational studies of such an inhomogeneous, complex random system are severely limited by the fact that observable quantities are integrals along the line-of-sight, so that many physically significant statistical features become hidden. When observed at a low resolution, the interstellar medium can be satisfactorily described in terms of Gaussian random fields, but recent observations have revealed a plethora of density structures in neutral hydrogen, mostly planar or filamentary (Heiles & Troland 2003; Makarenko et al. 2015; Wang et al. 2016, and references therein). Statistical analysis of such random fields cannot be restricted to the standard tools of the theory of Gaussian random functions (and related ones, such as log-normal and χ^2 functions), where the probability distribution and second-order correlation functions provide a complete description. However, correlation analysis remains an important first step, where the form of the correlation function, the correlation length (or time) and the mean-square variations of a variable are the most important quantities explored.

There are numerous and diverse estimates of the integral (correlation) scale of interstellar turbulence l_0 (see also Haverkorn & Spangler 2013). The autocorrelation function of the line-of-sight H I cloud velocities obtained in the Milky Way by Kaplan (1966) leads to l_0 ≃ 80pc. Lazaryan & Shutenkov (1990) found l_0 ≃ 50pc from the fluctuations in synchrotron intensity. Ohno & Shibata (1993) used differences in Faraday rotation between neighboring pulsars to obtain 10 < l_0 < 100pc. Minter & Spangler (1996) yielded l_0 ≃ 4pc from the structure functions of the variations in the Faraday rotation and emission measures across extended extragalactic radio sources. Structure functions of the Faraday rotation of extragalactic sources (Haverkorn et al. 2004, 2006, 2008) and their degree of depolarization (Haverkorn et al. 2008) give l_0 ≃ 1 pc in the Milky Way’s spiral arms. l_0 < 20pc was found by an analysis of low-frequency synchrotron intensity fluctuations from a large region of the Galactic disk by Iacobelli et al. (2013). In the Large Magellanic Cloud, the structure function of the Faraday rotation of more distant sources gave l_0 ≃ 90pc (Gaensler et al. 2005). In the galaxy M51 Fletcher et al. (2011) obtained l_0 ≃ 50pc from the depolarization of diffuse emission, whilst Houde et al. (2013) found l_0 ≃ 65pc from the dispersion of radio polarization angles. These estimates are strikingly different, perhaps not surprisingly. They have been obtained from diverse tracers, and it is...
not surprising that the correlation length of the gas velocities, Faraday rotation measure and synchrotron fluctuations differ (the latter being nonlinear functions of the fluctuating quantities). A relation between the correlation length of the product of random functions and those of the multipliers depends on their detailed statistical properties (e.g., §6.2 in Stepanov et al. 2014). Our aim here is to clarify this relation. This would be difficult to do with observational data, at least at present.

Interpretations of observations of polarized synchrotron emission and its Faraday rotation suggest that a significant fraction of the polarization may be due to anisotropy of the random magnetic field. The correlation between the mean Faraday rotation and its standard deviation along the Galactic disc, found by Brown & Taylor (2001), was the earliest indication of an anisotropic random field. Subsequent models of various components of Milky Way emission along the Galactic disk (Jaffe et al. 2010, 2011, 2013) and across the entire sky (Jansson & Farrar 2012a,b) required the inclusion of an anisotropic random magnetic field in order to fit the observations. In other galaxies, modeling of pre- and post-shock polarized emission in the barred galaxies NGC1097 and NGC1365 (Beck et al. 2005) and the spiral galaxy M51 (Fletcher et al. 2011), the dispersion of polarization angles in M51 (Houde et al. 2013), comparison of the observed polarized emission and Faraday rotation in M33 (Stepanov et al. 2014), and modeling depolarization in M51 (Shneider et al. 2014), have all indicated the presence of anisotropic random fields. Extracting the degree of anisotropy from the observations, though, is difficult.

In M51, Fletcher et al. (2011) estimate that the ratio of the standard deviations of the random magnetic field components in orthogonal directions is $\sigma_\|/\sigma_\perp \approx 2$ and Houde et al. (2013) obtained a ratio of correlation lengths along and perpendicular to the local mean-field direction of $l_\|/l_\perp \approx 1.8$. As with observational estimates for $l_0$, it is appropriate to carefully examine the possible anisotropy of the random magnetic field.

Simulations of the SN-regulated ISM have become sufficiently realistic to treat them as numerical experiments. It is then natural to use sufficiently realistic numerical models to address these questions before the more difficult observational exploration. We use such simulations, as detailed in Gent (2012) and Gent et al. (2013a,b), which have non-trivial magnetic fields generated by dynamo action, to clarify the correlation (and other statistical) properties of the multiphase ISM. In particular, we compare the autocorrelation and cross-correlation functions of the random (i.e., small-scale; see §2.2) velocity and magnetic fields and density fluctuations, as well as the Faraday depth and synchrotron intensity.

However complex, the simulations of the ISM can hardly be considered as trustworthy representations of the ISM in its whole complexity. Therefore, the goal of our analysis is not to achieve quantitative agreement with observations in every detail (although the general agreement is quite remarkable) but rather to identify those physical processes that shape the simulated ISM and are likely to be important in reality.

Turbulent flows are often represented in spectral space, in terms of the Fourier transforms of the physical variables. Such transforms are straightforward in infinite or periodic spaces. However, simulations of the ISM are performed in relatively small domains, only containing of order one thousand correlation cells, not simply-periodic because of the open (or similar) boundary conditions at the top and bottom of the domain, and statistically inhomogeneous because of the stratification (e.g. Korpi et al. 1999b,a; Gent et al. 2013a). Furthermore, it is difficult to estimate reliably the statistical uncertainty of the Fourier transforms.

We therefore proceed via correlation analysis (e.g., Monin & Yaglom 1975). For most of the work, we assume local isotropy in the horizontal $(xy)$ plane; this assumption is assessed in §5.

Correlation lengths obtained from comprehensive numerical simulations of the multi-phase ISM exhibit less diversity than the observational results. Joung & Mac Low (2006) obtain a gas density spectrum with a peak at 20pc, whereas most kinetic energy is contained at scales 20–40pc. Gent et al. (2013a) calculate $l_0 = 100$pc for the random velocity field in the mid-plane of the galaxy, also from hydrodynamic simulations. In the MHD simulations of de Avillez & Breitschwerdt (2007), $l_0 = 70$pc for the random velocity field. This scale fluctuates strongly with time. From correlation analysis of the vertical component of random velocity, Korpi et al. (1999b) obtained an estimate of $l_0 = 30$pc for the warm gas at all heights, whereas in the hot gas $l_0$ increases from 20pc in the mid-plane to 60pc at $|z| = 150$pc.

The paper is organized as follows. The simulations of the SN-driven ISM and averaging procedure used in our analysis are presented in §2. The spatial correlations of the random magnetic field, density and velocity are discussed in §3, whereas time correlations are the subject of §4. The anisotropy of the random magnetic field in the simulated ISM is estimated and interpreted in §5. The autocorrelation functions of such observable quantities as the Faraday depth and synchrotron intensity are obtained and discussed in §6. Our results are summarized in §7. Appendix A presents a comparison with the results obtained in a larger computational domain.

2. SIMULATIONS OF THE MULTI-PHASE ISM

We use our earlier simulations of the ISM based on the PENCIL CODE (http://pencil-code.nordita.org/), described in detail by Gent (2012) and Gent et al. (2013a). The simulations involve solving the full, compressible, non-ideal MHD equations with parameters generally typical of the solar neighborhood in a three-dimensional local Cartesian, shearing box with radial ($x$) and azimuthal ($y$) extents of $L_x = L_y = 1.024$ kpc and vertical ($z$) extent $L_z = 1.086$ kpc on either side of the mid-plane at $z = 0$. Our numerical resolution is $\Delta x = \Delta y = \Delta z = 4$ pc, using 256 grid points in $x$ and $y$ and 544 in $z$. Gent et al. (2013a) demonstrate that this resolution is sufficient to reproduce the known solutions for expanding SN remnants in the Sedov–Taylor and momentum-conserving phases.

The basic equations are mass conservation, the Navier–Stokes equation, the heat equation, and the induction equation, solved for mass density $\rho$, velocity $\mathbf{u}$, specific entropy $s$, and magnetic vector potential $\mathbf{A}$ (such that $\mathbf{B} = \nabla \times \mathbf{A}$).

The Navier–Stokes equation includes a fixed vertical gravity force that includes contributions from the stellar disk and spherical dark halo. The initial state is an approximate hydrostatic equilibrium. The Galactic differential rotation is modelled by a background shear flow $\mathbf{U} = (0, -q\Omega x, 0)$, where $q$ is the shear parameter and $\Omega$ is the Galactic angular velocity. Here we use $q = -1$, as in a flat rotation curve, and $\Omega = 25$ km s$^{-1}$ kpc$^{-1}$, as in the Solar neighbourhood.

The velocity $\mathbf{u}$ is the perturbation velocity in the rotating frame, that remains after the subtraction of the background shear flow from the total velocity. However, it still contains
a large-scale vertical component due to an outflow driven by the SN activity.

Both Type II and Type I SNe are included in the simulations. These differ in their vertical distribution and frequency only. The frequencies used correspond to those in the Solar neighborhood. We introduce Type II SNe at a mean rate, per unit surface area, of $\nu_1 = 25\;\text{kpc}^{-2}\;\text{Myr}^{-1}$. Type I SNe have a mean rate, per unit surface area, of $\nu_1 = 4\;\text{kpc}^{-2}\;\text{Myr}^{-1}$.

The SN sites are distributed randomly in the horizontal planes. Their vertical positions have Gaussian distributions with scale heights of $h_1 = 0.09\;\text{kpc}$ and $h_1 = 0.325\;\text{kpc}$ for SNII and SNI, respectively. No spatial clustering of the SNe is included. The thermal energy injected with each SN is $0.5 \times 10^{51}\;\text{erg}$. Injected velocity and the uneven density within each explosion site randomly adds kinetic energy with mean $0.4 \times 10^{51}\;\text{erg}$.

We include radiative cooling with a parameterized cooling function. For $T < 10^5\;\text{K}$, we adopt a power-law fit to the “standard equilibrium” pressure-density curve of Wolfire et al. (1995), as given in Sánchez-Salcedo et al. (2002). For $T > 10^5\;\text{K}$, we use the cooling function of Sarazin & White (1987). This cooling allows the ISM to separate into distinct hot, warm and cold phases identifiable as peaks in the joint probability distribution of gas in density and temperature.

Photoelectric heating is also included as in Wolfire et al. (1995). The heating decreases with $|z|$ on a length scale comparable to the scale height of the disk near the Sun. Shock-capturing kinetic, thermal and magnetic diffusivities (in addition to constant small background diffusivities), are included to resolve shock discontinuities and maintain numerical stability in the Navier–Stokes, heat and induction equations.

Periodic boundary conditions are used in $y$, and sheared-periodic boundary conditions in $x$ (considered in more detail in §2.4). Open boundary conditions, permitting outflow and inflow, are used at the vertical ($z$) boundaries. See Gent (2012) and Gent et al. (2013a,b) for further details on the boundary conditions used and on the other implementations described above.

Our analysis is based on 12 snapshots of the computational volume in the range $1.4 \leq t \leq 1.675\;\text{Gyr}$, by which time the system, including the large-scale magnetic field, has reached a statistically steady state. The interval between the snapshots, 25 Myr, is significantly longer than the correlation time of the random flow (see §4), and is sufficient for the snapshots to be considered statistically independent.

To test the influence of shear rate on the correlations, we also analyse data from a model with twice the rotation rate, as discussed in Gent et al. (2013b). We use 12 snapshots in the range $1.4 \leq t \leq 1.675\;\text{Gyr}$, again with a separation of 25 Myr, with the magnetic field saturated as for the main run. Any notable differences between the results for the different models will be reported throughout the text.

2.1. The multi-phase structure

The numerical model exhibits three distinct states of the gas corresponding to local maxima in the probability distribution function (PDF) of the specific entropy $s$. Gas parameters in those states are similar to the three main phases of the ISM. Following Gent et al. (2013a), the cold phase is defined as that having $s \leq 3.7 \times 10^7\;\text{erg}\;\text{K}^{-1}$, the hot phase has $s \geq 23.2 \times 10^7\;\text{erg}\;\text{K}^{-1}$, with the warm phase in between. The three phases have very different physical properties, including the random velocity and magnetic fields, as well as differing in their mean temperature and density. Therefore, our analysis is carried out for each phase separately. For this purpose, only grid points corresponding to a given phase are retained in the data cubes containing each physical variable, with the other points masked out. This allows us to do the averaging required in the computation of the structure functions over disjoint regions in the physical space.

2.2. Averaging procedure

Our analysis is conducted for the moduli of the random magnetic and velocity fields and the gas density fluctuations, denoted $b$, $u'$ and $n'$, respectively. The random velocity $u'$ should be carefully distinguished from the velocity perturbation $u$, defined as the deviation from the background large-scale shear flow, since the latter contains a systematic vertical velocity.

Since the mean vertical velocity and the large-scale magnetic field are not necessarily uniform across any horizontal plane, we do not use horizontal averages to define the mean magnetic field as is often done in the literature, but instead follow Gent et al. (2013b) and use Gaussian smoothing, within the framework of Germano (1992). The mean (large-scale) component of a random field $f$, averaged over a scale $\ell$ and denoted $\langle f \rangle_\ell$, is defined by a convolution with a Gaussian kernel $G_\ell(x)$,

$$\langle f \rangle_\ell(x) = \int_{-\ell/2}^{\ell/2} f(x') G_\ell(x-x')\,d^3x',$$

$$G_\ell(x) = (2\pi\ell^2)^{-3/2} \exp[-x^2/(2\ell^2)],$$

where integration is extended to the volume occupied by a given ISM phase or the total volume as appropriate. The random velocity is then $u' = u - \langle u \rangle$, and similarly for the magnetic field, $b = B - \langle B \rangle$, and the gas number density, $n' = n - \langle n \rangle$. Following Gent et al. (2013b), we use $\ell = 50\;\text{pc}$.

As discussed by Gent et al. (2013b), a significant fraction of the energy in the random field remains at length scales greater than $\ell$. To clarify the consequences of this, consider averaging a random field $f(x)$ in wave number ($k$) space, denoting $\tilde{f}(\mathbf{k})$ the Fourier transform of $f(x)$. By the convolution theorem, the mean field $\langle f \rangle_\ell(x)$ has the Fourier transform $\langle \tilde{f} \rangle_\ell(k) = \hat{\tilde{f}}(\mathbf{k}) \hat{G}_\ell(\mathbf{k})$, where $G_\ell(k)$ is the transform of the smoothing kernel. For the Gaussian kernel $G_\ell(x)$, we have $\hat{G}_\ell(k) = \exp(-\ell^2 k^2/2)$, so $\langle \tilde{f} \rangle_\ell(k) = \exp(-\ell^2 k^2/2) \hat{\tilde{f}}(\mathbf{k})$. Thus, for variations with wavenumber $k$ (and wavelength $\lambda = 2\pi/k$), a fraction $\exp(-\ell^2 k^2)$ of the original field energy is interpreted as that of the mean field, and the remainder goes into the random field. This fraction is half – i.e., the field energy is equally split between mean and random fields – at the wave number $k_{eq} = \sqrt{\ln 2}/\ell$, or the wavelength $\lambda_{eq} = 2\pi\ell/\sqrt{\ln 2} \approx 7.5\ell$. Thus, with $\ell = 50\;\text{pc}$, the field energy is equally split at the wavelength $\lambda_{eq} \approx 380\;\text{pc}$ between the mean and random parts. Variations with wavelength $\lambda < 380\;\text{pc}$ go predominantly into the random field, and increasingly so as $\lambda$ decreases; for features with $\lambda = 50\;\text{pc}$, only a fraction $\exp(-4\pi^2) \approx 10^{-17}$ of the energy goes into the mean field.

2.3. The structure and correlation functions

We start the calculations with the second-order structure functions $D(l)$, which are more robust than the correlation
functions, $C(l)$, with respect to errors (§13.1 in Monin & Yaglom 1975):

\[ D(l) = \langle [f(x + l) - f(x)]^2 \rangle, \]

where $x$ a given position in the $(x,y)$-plane and $l$ a horizontal offset with $l = |l|$. Analysis is restricted to horizontal planes with no offsets in the z-direction, because of the stratification in $z$.

Since we are dealing with periodic (or sheared periodic) functions in $x$ and $y$, the maximum offsets we can consider in the $x$ and $y$ directions are half the domain sizes in each direction. Hence, we consider offsets in the range $0 \leq l_x \leq L_x/2$, $0 \leq l_y \leq L_y/2$. Using $D(l)$, the autocorrelation function $C(l)$ is obtained as

\[ C(l) = 1 - \frac{D(l)}{2\sigma^2}, \]

where $2\sigma^2$ is the value of $D(l)$ at which the random function $f(x)$ is no longer correlated, and $\sigma$ is the dispersion (r.m.s. value) of $f(x)$. The choice of $2\sigma^2$ in a finite domain is not always obvious (see below). In terms of $C(l)$, the correlation length $l_0$ is defined as

\[ l_0 = \int_0^\infty C(l) \, dl. \]

The magnitude of the implied correlation length is very sensitive to the range of integration and to the behaviour of the correlation function at large $l$. An exponentially small tail in $C(l)$ can make a significant contribution to $l_0$.

To address this problem, we fit the structure functions obtained from equation (2) to one of the following analytic forms (as discussed below), thereby obtaining estimates of $\sigma^2$ and $L_0$ (and hence $l_0$):

\[ D(l) = 2\sigma^2 \left[ 1 - \exp \left( -\frac{l}{L_0} \right) \right], \quad l_0 = L_0, \]

\[ D(l) = 2\sigma^2 \left[ 1 - \exp \left( -\frac{l^2}{2L_0^2} \right) \right], \quad l_0 = \frac{\sqrt{\pi}}{2} L_0. \]

Since the governing equations contain second-order derivatives in spatial coordinates, the spatial variations must be smooth random functions of position, so that $dC/dl = 0$ at $l = 0$ for spatial correlations. However, the fact that only the first time derivatives appear in the governing equations implies that the time variations only needs to be continuous, so that $dC/d\tau \neq 0$ for $\tau = 0$ may be expected for time correlations (as considered in §4), with $\tau$ the time lag.

We indeed observe this difference in the computed structure and correlation functions, and use form in equation (5) for time correlations and equation (6) for spatial correlations. Some of the spatial autocorrelation functions discussed below (most notably those for the density fluctuations) exhibit an oscillatory behavior; in such cases, equation (6) is augmented to

\[ D(l) = 2\sigma^2 \left[ 1 - \exp \left( -\frac{l^2}{2L_0^2} \right) \cos(kl) \right], \]

with $k = a l + b$, where $a$ and $b$ are two additional parameters determined by the zeros in the correlation function.

Figure 1. Aligned domains of logarithm of gas number density ($\log n$), at $z = 2\text{pc}$, $t = 1.55\text{Gyr}$; (a) before and (b) after shifting the right-hand domain by $\delta y$ to account for the shearing boundary. The boundary between the two copies of the computational domain is here located at $x = 0$.

The correlation lengths $l_0$ are presented in Table 1. To confirm the importance of using fitted correlation functions, we also present in this table the correlation lengths $l_0$ obtained by integration of the directly calculated $C(l)$, over the range $0 \leq l \leq 500\text{pc}$. The values differ by up to a factor of 2, with the differences being greatest for density fluctuations (where the form in equation (7) was used); the agreement for random magnetic field and velocity (where the form in equation (6) was used) is closer.

To improve the reliability of our statistics, the averaging involved in the calculation of the structure functions is performed over 26 grid planes within layers at $|z| \leq 50\text{pc}$, $|z| - 0.4\text{kpc} \leq 50\text{pc}$ and $|z| + 0.4\text{kpc} \leq 50\text{pc}$ for each snapshot, and then the structure functions are further averaged over the snapshots. The uncertainty of the resulting values of the structure functions is rather small (of order $10^{-3}$ in terms of the relative error) because of the large number of data-point pairs available even at large values of $l$. The structure and correlation functions in figures below are shown with error bars representing not their uncertainty but the standard deviation of the individual measurements around the mean.

2.4. Accounting for shearing boundaries

When calculating the increments in the structure function, we use pairs of points separated by the periodic boundaries in $x$ and $y$. In the shearing box, the horizontal periodicity conditions (see Hawley et al. 1995) for a variable $f$ are

\[ f(x,y,z) = f(x + L_x,y - \delta y(t),z) \quad \text{(boundary in } x), \]

\[ f(x,y,z) = f(x,y + L_y,z) \quad \text{(boundary in } y), \]

where $\delta y(t) = \text{mod}(q\Omega L_y t, L_y)$ is the time-varying offset.
Table 1

The root-mean-square (rms) values of the fluctuations, their magnitude relative to the mean and correlation lengths of the fluctuations in gas density, speed and magnetic field for the warm gas at the mid-plane \( z = 0 \) and at \( |z| = 400 \) pc.

| \( |z| \) [pc] | Density fluctuations | Random speed | Random magnetic field |
|-------------|----------------------|--------------|----------------------|
|             | rms [cm\(^{-3}\)]   | \( l_0 \) [pc] | \( l_0 \) [pc] | rms [km/s] | \( l_0 \) [pc] | \( l_0 \) [pc] | rms [\( \mu \)G] | \( l_0 \) [pc] | \( l_0 \) [pc] |
| 0           | 0.306 ± 0.001       | 0.49 ± 0.03 | 53 ± 8      | 24 ± 1     | 8.10 ± 0.05 | 1.0 ± 0.1 | 60 ± 3     | 80 ± 1 | 0.587 ± 0.002 | 0.58 ± 0.04 | 64 ± 2 | 55 ± 1      |
| 400         | 0.604 ± 0.0001     | 0.49 ± 0.03 | 37 ± 2      | 27 ± 2     | 2.84 ± 0.01 | 0.49 ± 0.11 | 87 ± 3     | 81 ± 1 | 0.483 ± 0.001 | 0.39 ± 0.05 | 64 ± 2 | 55 ± 1      |

Note. — Two values of the correlation lengths are provided, \( l_0 \) obtained from a fitted form of the structure function as described in §2.3, and \( \tilde{l}_0 \) derived by integrating the calculated correlation function within the available range, \( 0 \leq l \leq 500 \) pc; the difference demonstrates how important is the fitting to obtain a reliable estimate of \( l_0 \).

Figure 2. (a) The structure function \( D(l) \) and (b) correlation functions \( C(l) \) for density fluctuations in the warm gas, averaged about \( z = -400 \) pc (blue, dashed), \( z = 0 \) pc (black, solid), and \( z = 400 \) pc (red, dash-dotted). The error bars denote the standard deviation of the individual measurements around their mean values, rather than the error of the mean value, as discussed in §2.3. For clarity, the error bars are only shown for every sixth bin in \( l \), and staggered between the different curves.

Figure 3. As in Fig. 2, but for the random speed in the warm gas.

Figure 4. As in Fig. 2, but for the modulus of the random magnetic field in the warm gas.

between the shearing boundaries in \( x \) (mapped to the range \( 0 \leq \delta y < L_y \)). In order to conveniently include pairs of points located on different sides of the the periodic boundary in \( x \), we extend the computational domain in the \( x \)-direction by its copy and shift it by \( \delta y(t) \) to remove the discontinuity between the two domains, as shown in Figure 1.

3. SPATIAL CORRELATIONS

As described above, we calculate the spatial structure- and correlation-functions for the random magnetic and velocity fields and the fluctuations in the gas number density separately for the warm and hot gas. The correlation functions are then used to estimate the correlation lengths of these variables. Spatial correlations of the Faraday depth and synchrotron emissivity are discussed in §6.

The results are shown in Table 1 and Figure 2 for the density fluctuations, Figure 3 for the random speed and Figure 4 for the magnitude of the random magnetic field. The structure functions used to obtain the autocorrelation functions are only shown in Figure 2a: those for the other variables have a similar form. The magnitudes of the fluctuations in the variables and their correlations lengths are discussed in the next two sections.

The uncertainties of the root-mean-square (rms) values of various variables and their correlations lengths given in Tables 1, 2 and 5 have been obtained as 95% confidence intervals from weighted least-squares fitting of equation (6), or for the gas density equation (7). The weights used are the uncertainties of the values of the correlation function rather than the standard deviations shown in the figures.

The uncertainties in the rms values and correlation lengths thus obtained are underestimates of the true uncertainty as they do not take into account any systematics errors, such as
those arising from the uncertain value of the computed structure functions at $l \to \infty$.

3.1. Magnitude of the fluctuations

The rms magnitudes of the fluctuations are shown in Table 1, together with the rms values of the relative fluctuations, $(\langle f'/(f) \rangle)^2$ for a variable $f$; we stress that the mean value $\langle f \rangle$ is a function of position. In the case of velocity fluctuations, the average velocity, $(\langle \mathbf{u} \rangle)_l = 0$, refers to the sheared frame, that is, includes the systematic outflow velocity, but not the overall rotation or the shear due to the galactic differential rotation.

In each phase, the standard deviation of the density fluctuations decreases with $|z|$ together with the average density. The relative magnitude of the fluctuations also decreases, but more slowly.

As shown in Fig. 2, density fluctuations are weakly anticorrelated in the range of scales $80 \leq l \leq 250 \, \text{pc}$ at each height, with the modulus of negativity for $C(l)$ significantly exceeding its uncertainty (about 0.002). Therefore, the rms value and correlation length of the density fluctuations has been obtained by fitting the form in equation (7) to the structure function. The parameters used in the cosine function were $k(l) = 0.07l + 0.28$ at $z = 0 \, \text{kpc}$ and $k(l) = 0.075l + 0.4$ at $|z| = 0.4 \, \text{kpc}$.

A possible cause of such anticorrelation may be random shock waves propagating through the ISM. Then the density fluctuations can be expected to be correlated within distances comparable to the shock thickness (about $5\Delta x = 20 \, \text{pc}$ in the simulations), whereas the anticorrelation arises from the systematic rarefaction associated with a shock front. Another effect that may contribute to such anticorrelation is the presence of quasi-spherical supernova remnants (as are clearly visible in Fig. 1), with gas density systematically lower than average within and around the bubbles and higher than average in their shells.

The rms random speed decreases with $|z|$ between $z = 0$ and $|z| = 400 \, \text{pc}$. This is understandable since the Type II supernovae, that drive most of the random flow, have a scale height of only 90 pc. At larger heights, the rms $u'$ is $5 \pm 1 \, \text{km} \, \text{s}^{-1}$ in the warm phase and $11 \pm 7 \, \text{km} \, \text{s}^{-1}$ in the hot gas at $|z| = 0.8 \, \text{kpc}$.

The magnitude of $\sigma_u$ in the simulations is below $\approx 5 \, \mu \text{G}$ observed near the Sun or in external galaxies (Beck 2016, and references therein). There could be several reasons for this, including the relatively low magnetic Reynolds numbers in the simulations reducing fluctuation dynamo efficiency, or an underestimated averaging scale $\ell$. However, it is evident from Figure 6 of Gent et al. (2013b), that its underestimation would not explain this. Applying horizontal averaging, which is analogous to extending $\ell$ to 1 kpc, yields an increase of only 50% in the saturated magnetic energy of the fluctuation field.

3.2. Correlation scales

The correlation length of the density fluctuations in the warm gas shown in Table 1 decreases with $z$, in the range $|z| \leq 400 \, \text{pc}$, in contrast to the correlation lengths of the velocity and magnetic fields.

In the simulations used here, shock-capturing diffusivities smooth shock fronts over five mesh points, i.e., 20 pc. This shock-capturing smoothing may affect the correlation lengths obtained, even though they are normally significantly larger than 20 pc. It may particularly affect the correlation length for the density fluctuations at $|z| = 400 \, \text{pc}$, which is only 37 pc.

The correlation length of the random velocity at the same height is significantly larger. The corresponding correlation length of the random magnetic field is intermediate between the two.

From the double rotation rate simulation, the results obtained for the correlation lengths and rms values are very similar to those in Table 1.

3.3. Taylor microscale

The Taylor microscale, $\lambda_t$, characterizes the behavior of the correlation function at small scales, $l \to 0$, and can be obtained by fitting the correlation function near the origin to the form

$$C(l) \simeq 1 - (l/\lambda)^2,$$

(§6.4 in Tennekes & Lumley 1972). The associated equality $dC/dl = 0$ at $l = 0$ holds for the correlation functions of smooth (differentiable) random fields (Monin & Yaglom 1975). In numerical simulations, where the solutions at the
C also obtained similar results in a model with doubled velocity effective Reynolds number in the simulations of order 20. We λ 60pc (Table 1) and ∆λx ∆x< 400pc (black, solid), and z = 0pc (red, dash-dotted). The resulting estimates of λ, shown in Table 2, satisfy the inequalities ∆λx < λ < l0, providing us some confidence in the estimates of the correlation lengths discussed above. For l0 = 60pc (Table 1) and λ = 40pc, we obtain an estimate of the effective Reynolds number in the simulations of order 20. We also obtained similar results in a model with doubled velocity shear.

3.4. Overall statistics and the cold and hot phases

Table 3

| l [pc] | Density fluctuations | | l [pc] | Random speed | | l [pc] | Random magnetic field | |
|-------|---------------------|---|-------|----------------|---|-------|---------------------|
| [cm⁻¹] | [cm⁻¹] | l [pc] | [cm⁻¹] | l [pc] | [µG] | l [pc] | [µG] |
| 0 | 0.305 ± 0.001 | 44 ± 2 | 29 ± 1 | 1.290 ± 0.03 | 74 ± 2 | 63 ± 1 | 0.382 ± 0.001 |
| 400 | 0.6004 ± 0.0001 | 37 ± 2 | 27 ± 2 | 3.65 ± 0.01 | 0.5 ± 0.1 | 117 ± 3 | 0.484 ± 0.001 |

The results presented above are for the warm gas. The data for the cold gas at offsets beyond l ∼ 100pc, the typical scale of the cold gas clouds, are scarce because the cold gas occupies a small fraction of the volume. Figure 6 only shows the cold phase results for the mid-plane, since the cold gas is concentrated there, and results outside this region cannot be statistically meaningful (see Gent 2012; Gent et al. 2013a). The structure functions for the hot phase fluctuate wildly and have large error bars (see Figure 7). This happens because the hot phase is extremely variable within the relatively small computational box that we have.

A separate analysis for each ISM phase, feasible with simulated data, may not be possible in observations. Therefore, we briefly discuss the statistical properties of the simulated ISM without separation by phase. The results are shown in Table 3.

As shown in Figure 8, the structure and correlation functions of magnetic fluctuations, b, for the whole ISM are almost identical to those in the warm phase. This is also true of the gas density fluctuations n. This similarity is reflected in the values of l, l′, σn, and σb. This is, of course, largely due to the large fractional volume of the warm phase. It is worth noting, however, that the density and magnetic field strength in the hot phase are both lower than in the warm phase. However, the values of σd and l′d for the whole ISM are
significantly higher than in the warm phase. The larger values of $\sigma_d$ for the whole ISM can be attributed to the contribution of the hot gas that has higher speed of sound and, correspondingly, higher random velocities.

4. TIME CORRELATION

Unlike the correlation lengths of various observable quantities in the ISM, their correlation times cannot be obtained from observations. Because of this, the eddy turnover time $\tau = l_0/\sigma_d$ is universally applied to interstellar turbulence. However, the dynamics of interstellar turbulence involves a range of physical processes having distinct time scales, which may make the eddy turnover time inappropriate as an estimate of the correlation time. Nonlinear Alfvén wave interactions, shock-wave turbulence and fluctuation dynamo action, among other phenomena, are likely to affect the correlation time and make it different for different variables.

Similarly to correlation lengths, the correlation times can be different in the warm and hot phases. However, this difference is harder to capture since each parcel of warm or hot gas moves around. Therefore, we can only obtain correlation times averaged over the ISM phases.

We consider arguably the most important of the time correlations, that of the random velocity. For this purpose, we use time series of the magnitude of the random velocity measured at an array of fixed points in 32 planes in $z$, separated by 64 pc; within each plane, there are 64 positions separated by 100 pc in $x$ or $y$.

From this data, we can calculate the temporal structure function, and then autocorrelation function $C(\tau)$, from which we obtain the correlation time $\tau_0$,

$$\tau_0 = \int_0^\infty C(\tau) \, d\tau. \quad (11)$$

We fit the form in equation (5) to $C(\tau)$ to estimate $\tau_0$.

The autocorrelation functions are shown in Figure 9 for four distances from the mid-plane, and the correlation times can be found in Table 4: $\tau_0 \approx 5$ Myr with little variation with $|z|$. Since the fractional volumes of the warm and hot gas vary significantly with $|z|$, this suggests that both phases have similar correlation times.

With the velocity correlation length and speed in the warm gas at $z = 0$ of 60 pc and 8 km s$^{-1}$, respectively (from Table 1) the kinematic time scale (‘eddy turnover time’) is of order $\tau_{\text{eddy}} = l_0/\sigma_d \simeq 8$ Myr. At $|z| = 400$ pc, we similarly have $\tau_{\text{eddy}} \simeq 30$ Myr in the warm gas.

According to the model of interstellar shock-wave turbulence of Bykov & Toptygin (1987), the separation of primary shock fronts driven by supernova explosions depends on their Mach number $M$ as

$$L_{\text{shock}} \simeq 4M^{4.5} \text{ pc}, \quad (12)$$

where the galactic supernova rate of 0.02 yr$^{-1}$ has been adopted. The primary shocks dominate over weaker secondary shocks for $M \gtrsim 1.2$, which leads to $L_{\text{shock}} \simeq 10$ pc. The corresponding time between crossings of a given position by shock fronts, which is expected to destroy time correlations, then follows as $\tau_{\text{shock}} = L_{\text{shock}}/c_s \simeq 1$ Myr, where $c_s = 10$ km s$^{-1}$ is the speed of sound in the warm gas.

In the simulations with double rotation rate, the velocity correlation rate and speed at the mid-plane in the warm phase change to 58 pc and 8 km s$^{-1}$, resulting in the eddy turnover time of $\tau_{\text{eddy}} \approx 7$ Myr, whereas $\tau_{\text{shock}}$ remains unchanged.

Since the estimate of $\tau_0$ that we have does not distinguish between the hot and warm phases, it depends on both the kinematic and shock-crossing time scales. All these time scales are of the same order of magnitude, so more careful estimates of the correlation time are required to clarify the physical nature of the time correlations in the simulated ISM.

It is plausible that the correlation time reflects both time scales and $\tau_0^{-1} \simeq \tau_{\text{eddy}}^{-1} + (1 - c)\tau_{\text{shock}}^{-1}$ with a certain constant $c$. With $\tau_0 = 5$ Myr, $\tau_e = 9$ Myr and $\tau_{\text{shock}} = 1$ Myr, we obtain $c = 0.1$, so the shock waves contribute about 10% to the random flow in this sense.

5. ANISOTROPY OF THE MAGNETIC FIELD

In the analysis above, we neglected any anisotropy of the random magnetic field in the horizontal planes. This is justified since, at the scales of interest (from a few parsecs to about 100 pc), the expected anisotropy is only moderate (see below). However, the anisotropy of magnetic fields is of high physical significance as it reflects the dynamics of MHD turbulence with and without a global mean magnetic field (Goldreich & Sridhar (1997); Brandenburg & Lazarian (2013) and references therein; see also Cho & Vishniac (2000); Cho & Lazarian (2002a, 2003b); Mallet et al. (2016); Oughton et al. (2016)), the effect of galactic differential rotation and compression of the random magnetic field in shocks. The anisotropy of interstellar magnetic fields can contribute significantly to the polarized radio emission of galaxies (e.g., Sokoloff et al. 1998; Beck 2016). In this section, using the structure and autocorrelation functions, we discuss individual components of the random magnetic field, $b = (b_x, b_y, b_z)$ de-
Since the vertical component of the random magnetic field is the strongest one.

An enhanced azimuthal ($y$) component is a result of the large-scale velocity shear due to differential rotation that produces $b_y$, so that $\partial b_y / \partial t \approx -q\Omega b_x$ and then (e.g., Stepanov et al. 2014)

$$b_{y0} \approx (1 - q\Omega\tau_0)b_{x0}. \quad (13)$$

For $q = -1$, $\Omega = 25 \text{ km} \text{s}^{-1} \text{kpc}^{-1}$ and $\tau_0 = 5 \text{ Myr}$, this yields $b_{y0}/b_{x0} \approx 1.2$–1.3, in agreement with the estimates of Table 5 at $z = 0$.

The vertical component of the magnetic field is similarly enhanced beyond isotropy due to the stretching of the horizontal magnetic field by vertical velocity $u_y$ that varies at a scale $l_y$ and yet has a mean part $\mathcal{P}_y \approx 2 \text{ km} \text{s}^{-1}$ at $|z| \lesssim 200 \text{ pc}$: $\partial b_y / \partial t \approx \beta y_0\partial u_y / \partial x + b_x\partial u_y / \partial y$. Unlike the stretching of the radial magnetic field by the large-scale velocity shear, this is a random process, so the rms vertical magnetic field grows as $t^{1/2}$. With the radial field $b_r$ representing the isotropic background, this leads to the estimate

$$\frac{b_{y0}}{b_{x0}} \approx \left[ 1 + \frac{\tau_0\mathcal{P}_y}{l_0} \left( 1 + \frac{b_{y0}^2}{b_{x0}^2} \right) \right]^{1/2} \approx 1.2,$$

in a reasonable agreement with the estimates of Table 5. Since the vertical component of the random magnetic field is produced from both of its horizontal components, the $z$-component is the strongest one.

An important radio astronomical consequence of the magnetic anisotropy is polarization of the synchrotron emission. If our simulation domain was observed from the top or bottom (i.e., along the $z$ direction) the observed degree of polarization due to the random magnetic field alone would be (Lai 1981; Sokoloff et al. 1998, 1999)

$$p = p_0 \frac{|b_{y0}^2 - b_{x0}^2|}{b_{x0}^2 + b_{y0}^2} \approx 0.15,$$

where $p_0 = 0.7$ is the maximum intrinsic degree of polarization, and we have neglected, for the sake of the argument, both depolarization effects and the average magnetic field. Such a degree of polarization is comparable to that observed in spiral galaxies, suggesting that the anisotropy of the interstellar random magnetic fields needs to be allowed for in the interpretations of radio polarization observations of spiral galaxies (Beck 2016).

The correlation lengths of the magnetic field components are given in Table 5 (for comparison with Table 1). Because of the stretching of radial magnetic field by differential rotation that produces a stronger azimuthal field, we might expect the azimuthal correlation length to be larger than the radial one (Moffatt 1967; Terry 2000), contrary to the results in Table 5, where the correlation lengths for $b_z$ and $b_x$ are of similar magnitude. However, the correlation lengths were calculated using isotropic horizontal position lags, whereas azimuthal ($y$) and radial ($x$) lags should be considered separately to detect the expected difference in the correlation lengths in the two directions. Such a refined calculation requires a larger data domain to provide sufficient statistics. Houde et al. (2013) find that $l_{y0} \approx 1.8l_{x0}$ for the random magnetic field, i.e., the magnetic correlation length approximately along the mean-field direction ($y$ in our case) is about twice that in the perpendicular direction, and this ratio is similar to the ratio of $b_{y0}/b_{x0}$ found by these authors from depolarization of the synchrotron emission. The vertical magnetic field component has significant anticorrelation at $l \approx 100 \text{ pc}$, shown in Figure 10, which results in very different values of $l_0$ and $l_y$, similar to $n'$. As shown in Figure 11, individual components of the random magnetic field vary differently with $|z|$. As with the mean magnetic field, the rms means first increase with distance from the mid-plane until $|z| \approx 200 \text{ pc}$, and only then decrease. As

### Table 5

| $|z| \text{ [pc]}$ | $b_{x0}$ [$\mu G$] | $b_{y0}$ [pc] | $l_0$ [pc] |
|-----------------|-----------------|-----------------|-----------------|
| 0               | 0               | 0               | 0               |
| 400             | 549 ± 1         | 432 ± 1         | 46 ± 2          |
| 0               | 400             | 51 ± 3          | 35 ± 1          |
| 0               | 0               | 52 ± 1          | 32 ± 1          |

Note. — The correlation lengths $l_0$, using Eq. (7), and $l_{x0}$ are calculated as in Table 1.
suggested above, both \( b_x \) and \( b_z \) are enhanced, in comparison with \( b_y \), by the horizontal velocity shear and random vertical flows, respectively; correspondingly, \( b_{0y} \) and \( b_{0z} \) increase with \( |z| \) faster than \( b_{0x} \) at \( |z| \lesssim 200\,\text{pc} \), but then decrease with \( |z| \) following the decrease in \( b_{0x} \). At \( |z| \gtrsim 300\,\text{pc} \), each component of \( \mathbf{b} \) decreases nearly exponentially with the scale height of about \( 450\,\text{pc} \).

Simulations with double rotation rate produce similar results.

6. OBSERVABLE QUANTITIES

The main observational tools employed in the analysis of interstellar MHD turbulence are Faraday rotation and synchrotron emission, both total and polarized. Their statistical properties and their relation to the underlying random distributions of magnetic fields, gas density and cosmic rays have received significant attention, both observationally and theoretically (see references in §1). Here we discuss correlation properties of the observable quantities in the simulated ISM. Given that magnetic field and gas density can have different distributions, both total and polarized. Their statistical properties and their relation to the underlying random distribution can have different functions, and can be correlated with each other (Beck et al. 2003), statistical properties of the observable quantities are difficult to predict with confidence.

Both Faraday rotation and synchrotron emission depend on the relative orientation of the large-scale magnetic field and the line of sight. The mean magnetic field in the simulations used here is predominantly horizontal and its \( y \)-component is the strongest (Gent et al. 2013a,b). Exploring the observational appearance of the simulated volume from various vantage points will be our goal elsewhere; here we only discuss the properties of fluctuations in Faraday rotation and synchrotron emission using just one direction of ‘observation’.

6.1. Faraday depth

The Faraday depth of a magneto-ionic region is an integral along the line of sight, assumed here to be along the \( z \)-direction for convenience:

\[
\phi(x, y) = 0.81 \int_{-L_z}^{L_z} n_e B_z \, dz \, \text{rad m}^{-2},
\]

where \( n_e \) is the number density of thermal electrons in \( \text{cm}^{-3} \), \( B_z \) is the line-of-sight component of magnetic field in \( \mu\text{G} \), distance \( z \) is in \( \text{pc} \), and \( L_z \) is the half-size of the computational domain along \( z \). Our simulations do not include gas ionization and only provide total gas density \( n \). Since interstellar plasmas can be far from ionization equilibrium (de Avillez & Breitschwerdt 2012a,b), we obtain thermal electron density from a heuristic relation that ensures that the mean electron number density is about \( 0.05\,\text{cm}^{-3} \) and the gas is fully ionized at \( T \gtrsim 10^6 \,\text{K} \):

\[
n_e = n \left[ \frac{\arctan(T / 10^6 \,\text{K} - 10)}{\pi} + \frac{1}{2} \right].
\]

Since observations do not distinguish between different ISM phases, the Faraday depth has been computed for the whole computational domain.

The autocorrelation function of the Faraday depth is shown in Figure 12. Its correlation length, \( l_\phi = 122 \pm 12\,\text{pc} \) is significantly greater than the correlation length of electron density, \( 60\,\text{pc} \) at the midplane increasing to \( 80\,\text{pc} \) at \( |z| = 800\,\text{pc} \) (Table 6), and the vertical random magnetic field, \( 60\,\text{pc} \) (Table 5). We note that the mean component of \( B_z \) is negligible, so that the mean value of the Faraday depth is close to zero, \( \langle \phi \rangle = 2.88 \pm 0.42 \,\text{rad m}^{-2} \).

As discussed by Beck et al. (2003), the magnitude of Faraday rotation depends on the correlation between magnetic field and thermal electron density. To clarify their relation in our simulations, we computed the cross-correlation coefficient between \( n_e \) and \( B_z \) separately for the warm and hot gas:

\[
r = \frac{(n_e - \bar{n}_e)(B_z - \bar{B}_z)}{(n_e - \bar{n}_e)^{1/2}(B_z - \bar{B}_z)^{1/2}}
\]

where the overbar denotes an average taken over the volume occupied by the phase. The results, averaged over the snapshots, confidently suggest that the two variables are uncorrelated: \( r = 0.02 \pm 0.02 \) in the warm gas and \( 0.07 \pm 0.04 \) in the hot phase.

6.2. Synchrotron intensity

Statistical properties of the synchrotron intensity are sensitive to the relation between the distributions of cosmic ray electrons, \( n_{\text{cr}} \), and magnetic field. Cosmic rays (Berezinskii et al. 1990) have a high diffusivity of order \( 3 \times 10^{28} \,\text{cm}^2\,\text{s}^{-1} \), so their diffusion length over the confinement time of \( 10^6 \,\text{yr} \) is of order 1 kpc. Thus, it can be expected that cosmic rays are distributed much more homogeneously than magnetic fields, but the assumption of a local energy equipartition (or pressure balance) between cosmic rays and magnetic fields is often used in interpretations of synchrotron observations (e.g., Beck & Krause 2005). We note that analysis of synchrotron fluctuations in spiral galaxies suggest that cosmic ray electrons and magnetic fields can be slightly anti-correlated (Stepanov et al. 2014). Fluctuations of synchrotron intensity can provide information about interstellar turbulence (Lazarian & Pogosyan 2012, 2016). Here we discuss the synchrotron intensity fluctuations implied by our ISM simulations.

| \(|z|\) [pc] | \(\text{rms} \) [\(\text{cm}^{-3}\)] | \(l_\phi\) [pc] |
|---|---|---|
| 0 | 0.2500 ± 0.0007 | 59 ± 3 |
| 200 | 0.1665 ± 0.0008 | 61 ± 5 |
| 400 | 0.0530 ± 0.0003 | 80 ± 6 |
| 600 | 0.0208 ± 0.0002 | 93 ± 8 |
| 800 | 0.0083 ± 0.0001 | 83 ± 7 |

Figure 12. The autocorrelation function of the Faraday depth \( \phi(x, y) \). The error bars represent the scatter of the data points around the mean values shown with solid line.

Table 6

The rms values and the correlation lengths for electron density \( n_e \).
The synchrotron intensity, in arbitrary units, is obtained by integration along the z-axis (so that the mean magnetic field is mostly perpendicular to the line of sight),

\[ I(x, y) = \int_{-L_z}^{L_z} n_{\text{cr}} \left( B_x^2 + B_y^2 \right) dx, \]

using two alternative assumptions about cosmic ray distribution \( n_{\text{cr}} \):

- \( n_{\text{cr}} = \text{const} \),
- \( n_{\text{cr}} \propto B^2 \).

As with the Faraday depth, we do not consider other lines of sight through the computational domain.

The Stokes parameters, at wavelengths short enough that Faraday rotation is negligible, are similarly obtained as

\[ Q(x, y) = \int_{-L_z}^{L_z} \cos(2\psi_0) n_{\text{cr}} \left( B_x^2 + B_y^2 \right) dx, \]
\[ U(x, y) = \int_{-L_z}^{L_z} \sin(2\psi_0) n_{\text{cr}} \left( B_x^2 + B_y^2 \right) dx, \]

where \( \psi_0(x) \) is the intrinsic polarization angle perpendicular to the local magnetic field in the (xy)-plane, calculated as \( \psi_0 = \pi/2 + \arctan(B_y/B_x) \). The polarized intensity follows as

\[ P(x, y) = \sqrt{Q^2 + U^2}. \]
fluctuations (the Fourier transforms of the correlation function) as suggested by Lazarian & Pogosyan (2012, 2016) and Lee et al. (2016). This problem may not be evident when power spectra are considered because it is difficult to estimate their statistical accuracy. However, correlation analysis, with due attention to the errors, makes the problem evident.

7. DISCUSSION

We have performed detailed correlation analysis of the random physical fields in extensive ISM simulations, focusing mainly on the warm gas since it occupies a larger part of the volume. Statistical properties of the fluctuations in the gas properties are strongly non-Gaussian because of widespread filamentary and planar, small-scale structures. Such features cannot be captured by second-order correlation functions (or their equivalent, power spectra) and require other tools sensitive to all statistical moments of the random field, such as Minkowski functionals (e.g., Wilkin et al. 2007; Makarenko et al. 2015, and references therein) and topological data analysis (Adler et al. 2010; Edelsbrunner 2014). However, careful correlation analysis remains a necessary first step in the exploration of statistical properties of random fields.

There are two difficulties in correlation analysis (and its equivalent, power spectrum analysis) that deserve special attention as they also occur in any exploration of either simulated or observational data. Correlation analysis is only meaningful when applied to a random distribution. Therefore, random fluctuations in physical parameters need to be isolated first by subtracting their averaged distributions. Averaging is equivalent, power spectrum analysis) that deserve special attention as they also occur in any exploration of either simulated or observational data. Correlation analysis is only meaningful when applied to a random distribution. Therefore, random fluctuations in physical parameters need to be isolated first by subtracting their averaged distributions. Averaging is straightforward in infinite domains with statistically homogeneous fluctuations. However, in reality the domain can contain only a modest number of correlation volumes, and the mean distributions of physical variables are not necessarily uniform or describable via a simple trend. We obtain the averaged distributions using Gaussian smoothing at a scale (half-width of the Gaussian window) of 50pc chosen carefully as in Gent et al. (2013b) (see Section 2.2). Simpler procedures, for example using a uniform mean value at a given $z$, distort the results because of the contamination of the structure and correlation functions by systematic and complicated non-random trends. In particular, the values of correlations lengths obtained under the assumption of horizontally uniform mean values are unphysically large, exceeding 200pc.

Even with a correlation lengths $l_0$ of less than 100pc, the finite size of the domain (of order 1kpc$^3$ in our case) can significantly affect the estimated values of $l_0$, as the integration in Equation (4) extends to infinity. We resolve this problem by fitting the measured correlation functions with physically motivated forms, which can then be integrated over an infinite range. The difference between the correlations lengths obtained with and without this fitting can be as large as a factor of two.

Given the complex structure of the simulated ISM, it is not surprising that different physical variables have different correlation functions and different correlation lengths $l_0$, as shown in Table 1. The observational estimates available for the correlations lengths in the ISM provide a wide range of values depending on the quantity observed. Conclusive comparison with observations requires detailed knowledge of the statistical properties of the random fields involved and their cross-correlations (Stepanov et al. 2014). Interstellar turbulence cannot be characterized by a single correlation length.

We have estimated the correlation time of the velocity fluctuations $\tau_0$, in the simulations used here, $\tau_0 \approx 5$ Myr is close to both the eddy turnover time, $\tau_{\text{eddy}} \approx 8$ Myr and the estimated time interval between the passage of shock fronts through a given position, $\tau_{\text{shock}} \approx 1$ Myr. The correlation time is likely to be sensitive to the supernova rate (and then, star formation rate) and may be closer to $\tau_{\text{shock}}$ when the supernova rate is higher. Further calculations with varying supernova rates are needed to explore under what conditions either physical process dominates the correlation time.

The random magnetic field is noticeably anisotropic, with larger rms values for azimuthal ($\theta$) and vertical ($z$) components in comparison to the radial ($x$) component, with $b_x$ the strongest component. The enhanced $\theta$-component is produced by the action of the large-scale velocity shear on the radial turbulent magnetic field $b_x$, with the enhanced $z$ component produced by stretching of the horizontal magnetic field by the random part of the vertical velocity $u_z$. From the rms values of $b_x$ and $b_y$, we estimate a degree of polarization of $p \approx 0.15$ that may be produced by the magnetic anisotropy.

We also performed correlation analysis of the Faraday depth along the vertical direction through the computational domain. Its correlation scale, 120pc, is significantly larger than the correlation scales of electron density (60–90pc) and of vertical magnetic field (60pc). This suggests that there is no simple and universal relationship between the correlation scales of electron density, vertical magnetic field and Faraday depth.

Analysis of the total and polarized synchrotron intensities is hampered by a rapid increase of the scatter of data points around the average contributions to the structure and correlation functions. This difficulty is evident in the correlation analysis but would not be apparent in the power spectra, where statistical errors are difficult to estimate.

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APPENDIX

A. COMPARISON WITH LARGER DOMAIN

The computational domain used to obtain our results, about $1 \times 1 \times 2$ kpc$^3$, contains only about $10^3$ correlation cells and, in addition, may be too small to accommodate the most rapidly growing mode of the large-scale magnetic field. The large-scale dynamo remains in its kinematic stage in the larger domain, but otherwise the simulation has achieved a statistically steady state. Therefore, we verify the results using similar simulations in a larger domain, approximately $1.352 \times 1.532 \times 2.556$ kpc$^3$ in size. The velocity shear rate is that of the Solar neighborhood, $q = -1$, and we analyze data from 12 snapshots in the range $0.336 \leq t \leq 0.6$ Gyr, with a separation of 24 Myr.

The results from the larger domain are compared with those obtained from the kinematic stage of the large-scale dynamo in the main run discussed in the text. We use data from 21 snapshots in the range $0.4 \leq t \leq 0.61$ Gyr, with a separation of 10 Myr.

We find very similar correlations in $b$ between the two runs (see Figure 15 and Table 7), but there are more significant
Table 7
Root-mean-square values and correlation lengths of the random magnetic and velocity fields in the standard and larger domains, for simulations in the kinematic stage of dynamo action.

<table>
<thead>
<tr>
<th>z [pc]</th>
<th>rms fluctuations</th>
<th>l₀ [pc]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Standard domain</td>
</tr>
<tr>
<td>0</td>
<td>0.196 ± 0.001</td>
<td>0.166 ± 0.001</td>
</tr>
<tr>
<td>400</td>
<td>0.161 ± 0.001</td>
<td>0.123 ± 0.001</td>
</tr>
<tr>
<td>Random Magnetic field [μG]</td>
<td>33.4 ± 0.2</td>
<td>16.2 ± 0.1</td>
</tr>
<tr>
<td>Random speed [km s⁻¹]</td>
<td>11.3 ± 0.1</td>
<td>3.12 ± 0.04</td>
</tr>
</tbody>
</table>

Figure 15. Comparison of structure functions for random magnetic field strength b, for (a) the standard domain and (b) the larger domain; averaged about z = −400 pc (blue), 0 pc (black), and 400 pc (red). Both plots use data from the kinematic phases of the simulations.

Figure 16. Comparison of structure functions for random speed u′, comparing domain size as in Figure 15.

differences for u′ (see Figure 16 and Table 7; the latter also gives comparable statistics for a similar kinematic state in the standard domain). The correlation lengths of u′ are actually smaller for the larger domain, so the difference does not simply result from velocity structures having been restricted in size. In light of the differences noted above, further simulations are needed before a direct comparison can be made.

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