

## Motivation



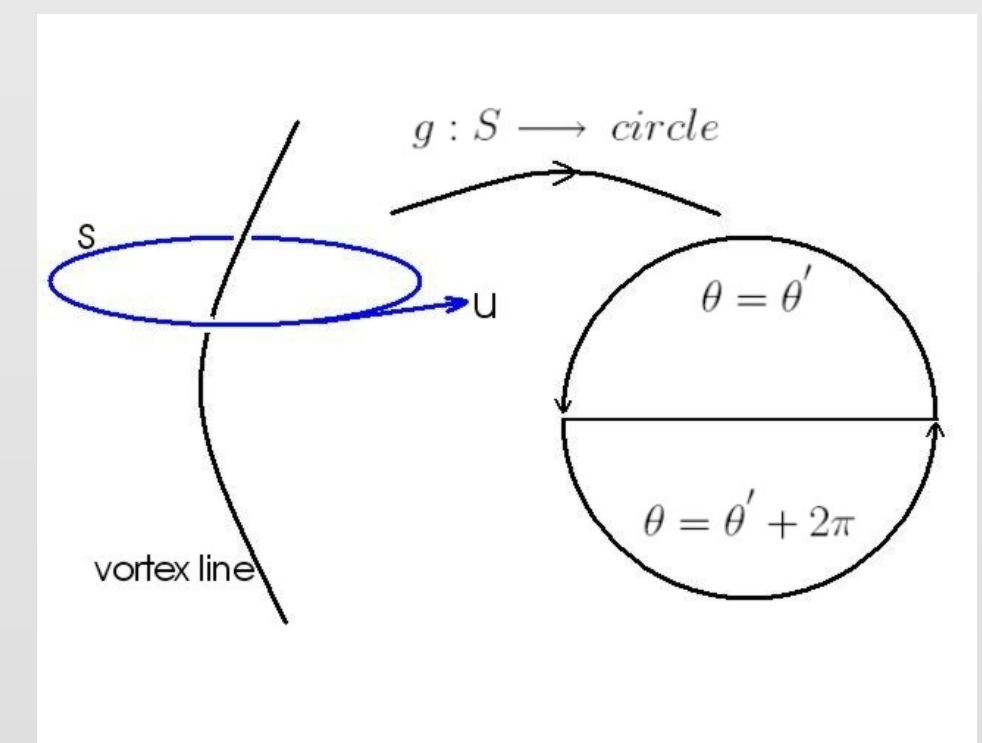
The mathematical object gerbe derives its name from the French 'gerbe' meaning wheat-sheaf. The interlacing of kernels on an ear of wheat is analogous to the intersection of related topological spaces being considered in the mathematical context. The electromagnetic field strength is a trivial example of a gerbe. We shall consider a less trivial example, the helicity of a magnetic field configuration, using a simple gerbe definition.

## Definition Of The Gerbe

Gerbes involve circle valued function mappings  $f : X \rightarrow S^1$ .  $S^1$  is the set of unit circles.

$U_\alpha$  an open cover of  $X$ ,  $f = \bigcup g_\alpha$

s.t.  $g_\alpha : U_\alpha \rightarrow S^1$  & if  $U_\alpha \cap U_\beta \neq \emptyset$   $g_\beta g_\alpha^{-1} = 1$   
A simple example of a gerbe is the line vortex



A wave function of a vortex fluid

$$\psi(\mathbf{x}, t) = \rho e^{i\varphi} = \rho g,$$

where  $g$  maps  $S$  to  $S^1$ . The fluid velocity  $\mathbf{u} \propto d\varphi = -ig^{-1}dg$ , so

$$\int_S \mathbf{u} = k \int_S d\varphi = k \times 2\pi n.$$

If  $S$  is contractible then  $n = 0$ . In superfluids quantum jumps can occur due to discontinuity on the vortex line.

We define gerbes with connection a set

$$\{g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha\},$$

where  $B_\alpha$  is defined on the open cover  $U_\alpha$  s.t.

$$B_\alpha - B_\beta = dA_{\alpha\beta} \quad \& \quad A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = ig_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}$$

If  $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \neq \emptyset \Rightarrow g_{\beta\gamma\delta} g_{\alpha\gamma\delta}^{-1} g_{\alpha\beta\delta} g_{\alpha\beta\gamma}^{-1} = 1.$

## The Calculus Of Differential Forms

We denote vector field components  $A_j$ , on a differentiable manifold  $X_n$  with local coordinates  $x^j$ . The point displacement has components  $dx^j$ . We define a 1-form

$$\omega = A_j dx^j,$$

The exterior product (wedge  $\wedge$ ) of 1-forms  $\omega$  and  $\lambda = B_h dx^h$  is a 2-form

$$A_j dx^j \wedge B_h dx^h = \frac{1}{2}(A_j B_h - A_h B_j) dx^j \wedge dx^h$$

A coordinate transformation  $\hat{x}^j = \hat{x}^j(x^h)$  is

$$d\hat{x}^j = \frac{\partial \hat{x}^j}{\partial x^l} dx^l \Rightarrow d\hat{x}^j \wedge d\hat{x}^i = \frac{1}{2} \left( \frac{\partial \hat{x}^j}{\partial x^l} \frac{\partial \hat{x}^i}{\partial x^k} - \frac{\partial \hat{x}^j}{\partial x^k} \frac{\partial \hat{x}^i}{\partial x^l} \right) dx^l \wedge dx^k$$

The exterior derivative of  $\omega$  is defined

$$d\omega = \frac{1}{2} \left( \frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} \right) dx^k \wedge dx^j$$

## Poincaré's Lemma And Stokes' Theorem

**Lemma:** If  $\omega$  is a  $p$ -form and there is a  $(p-1)$ -form  $\eta$  such that  $\omega = d\eta$  then  $\omega$  is said to be exact.

If  $X$  is contractible,  $\eta$  is global

$$\Rightarrow d\omega = d(d\eta) = \frac{1}{2} \left( \frac{\partial^2 A_j}{\partial x^k \partial x^j} - \frac{\partial^2 A_k}{\partial x^j \partial x^k} \right) dx^k \wedge dx^j = 0.$$

**Theorem:**  $G$  a smooth, analytic & simply connected  $(p+1)$ -dimensional region of  $X_n$  with  $\partial G$  a piece-wise smooth closed  $p$ -dimensional boundary

$$\int_G d\omega = \int_{\partial G} \omega$$

If the region  $G$  is globally exact then  $g_\beta g_\alpha^{-1} = 1$  everywhere, otherwise  $g_\beta g_\alpha^{-1} \neq 1$  possible for  $(U_\alpha \cap U_\beta)^C$ .

## Maxwell's Equations

$$\nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

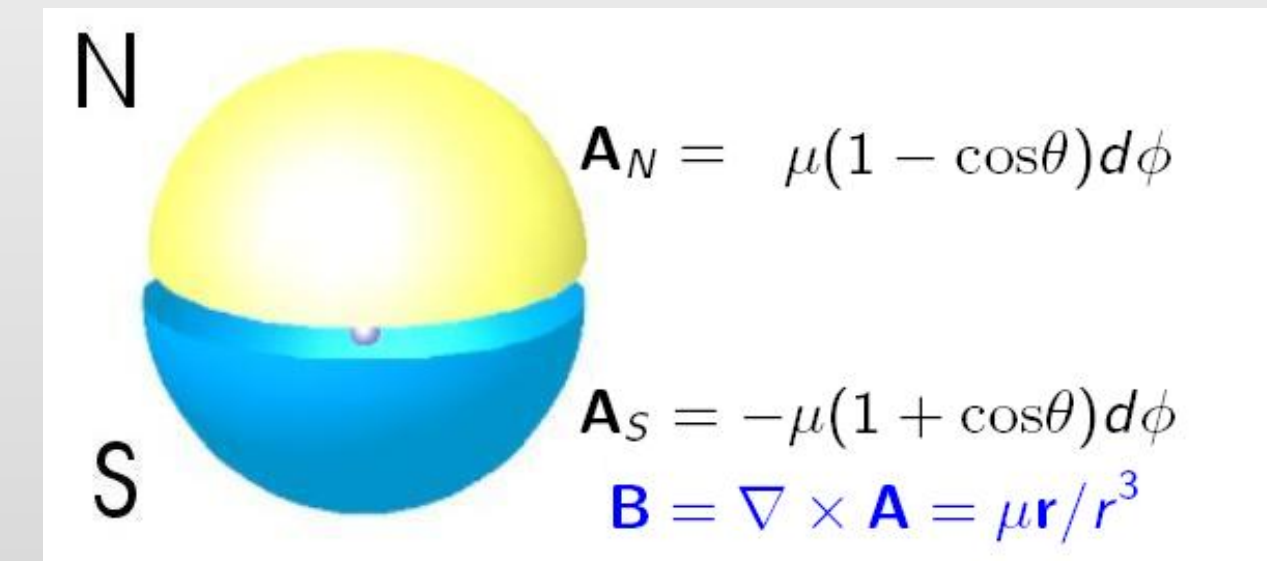
$$\nabla \cdot \mathbf{E} = \rho, \quad \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mathbf{j}$$

Maxwell's Equations can be condensed in 4-dimensions to the 3-form  $d(*\mathbf{F}) = 0$ .  $*$  denotes the Hodge\* operator, which is used to transform the exact  $d\mathbf{F} \equiv 0$  into an equation with a non-trivial solution. We solve this for a toroidal field.

## The Dirac Monopole

Dirac devised the monopole theoretically to reconcile Maxwell's equations so the point magnetic force  $\mu$  corresponds to the point charge  $e$ .

The magnetic potentials  $\mathbf{A}_N$  and  $\mathbf{A}_S$  induce the same field  $\mathbf{B} = \mu\mathbf{r}/r^3$ , each singular on the opposite z-axis. Combining both defined in opposite hemispheres we obtain two analytic simply connected fields excluding  $r = 0$ .



The intersection of the fields on the equatorial surfaces forms part of a 0-gerbe. Dirac proved the magnetic flux strength  $\Phi$  must be quantised

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = 4\pi\mu \equiv \frac{e}{hc} \mathbf{A}_N - \frac{e}{hc} \mathbf{A}_S = -ig^{-1}dg \Rightarrow g = e^{2i\mu e\phi/\hbar c} \Rightarrow \mu = \frac{\hbar c n}{2e}$$

## A New Solution Of Maxwell's Equations

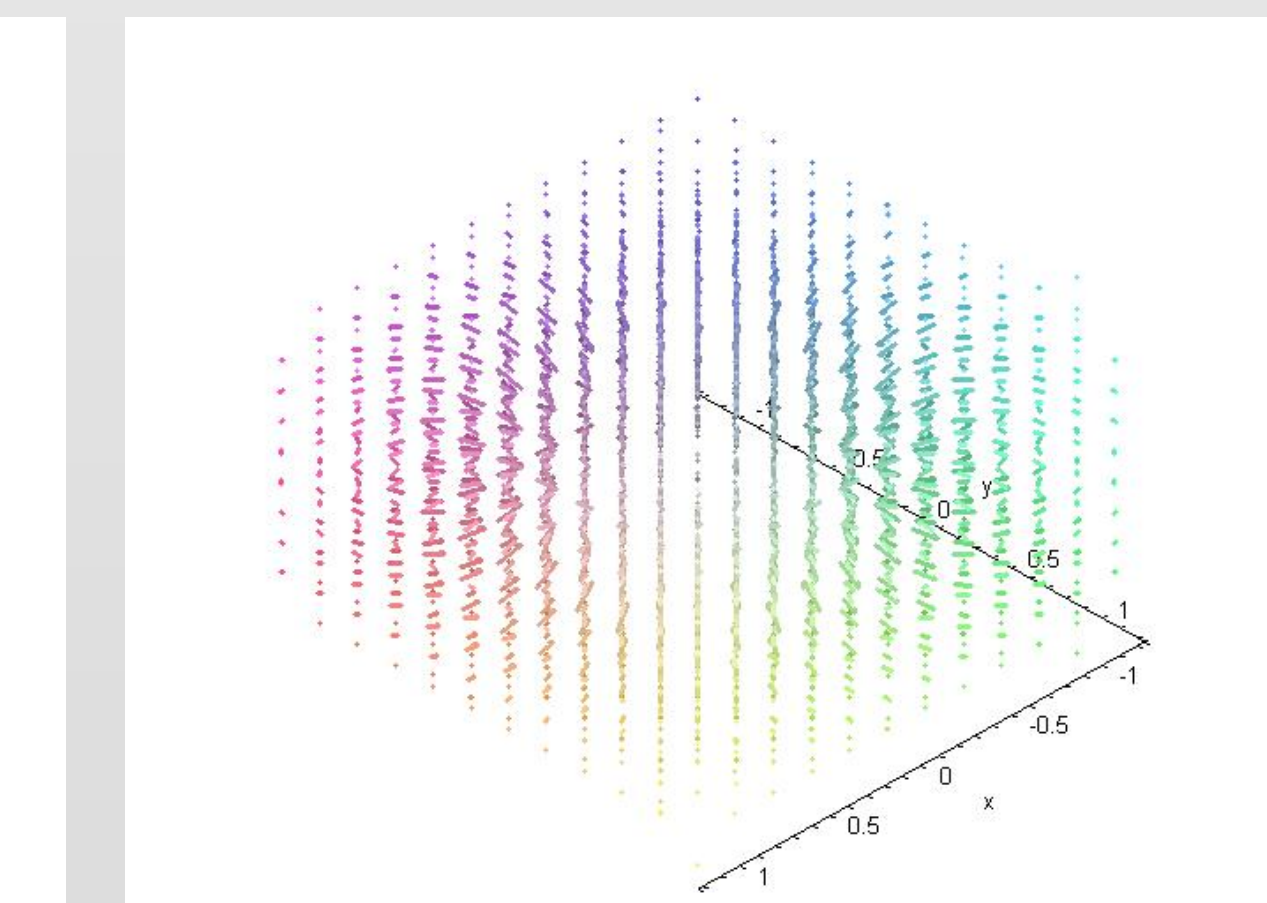
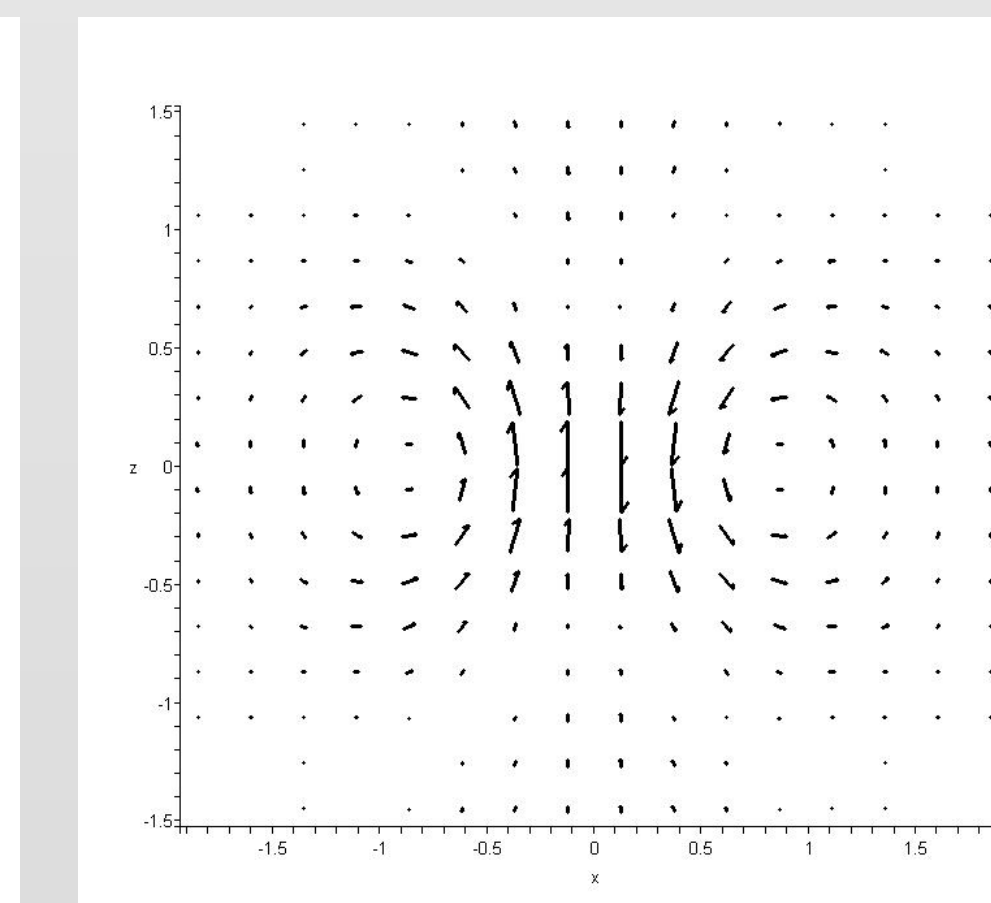
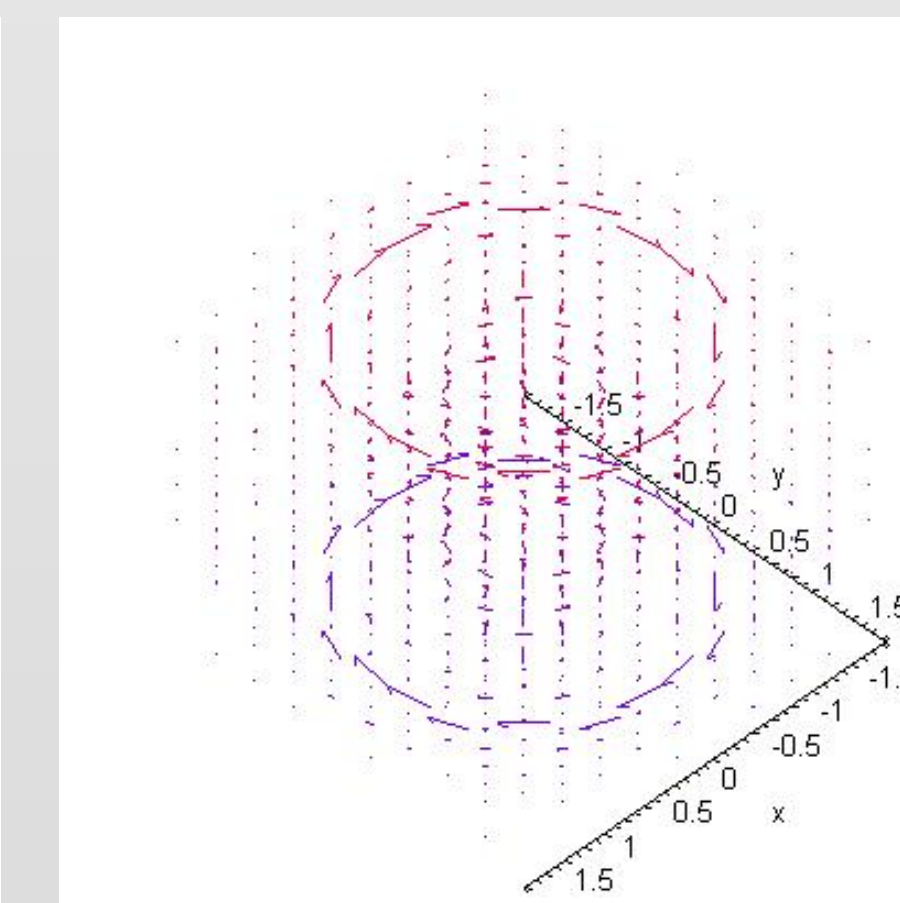
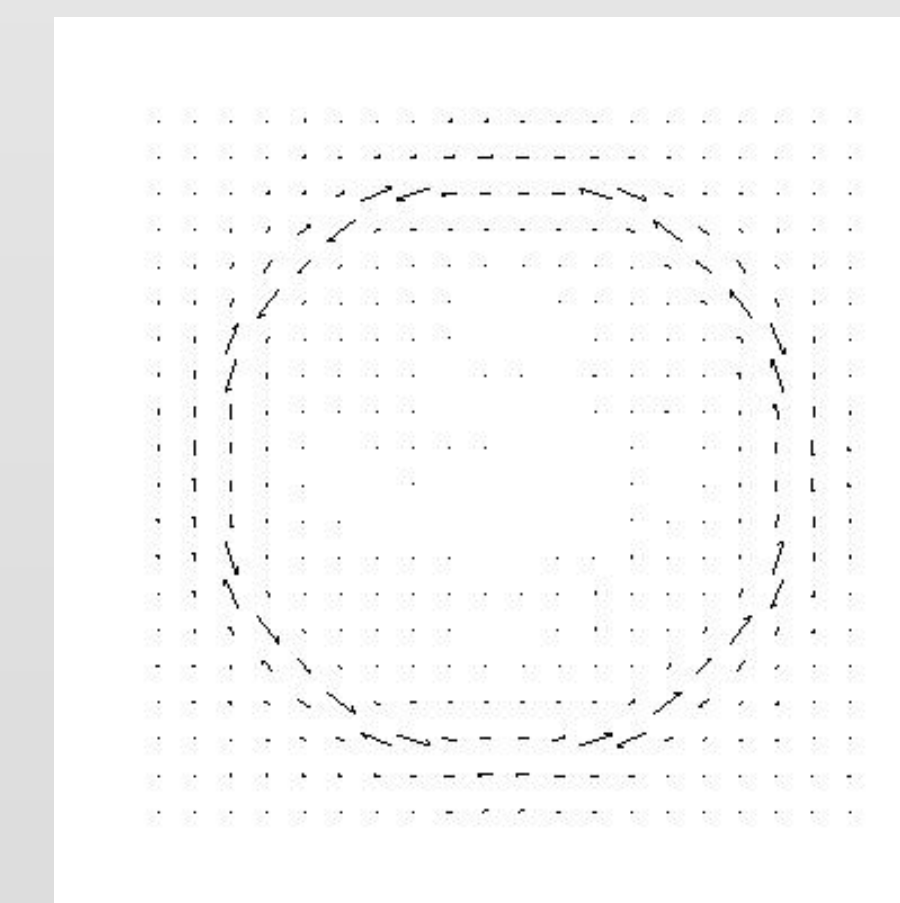
We consider magnetic potential  $\mathbf{A} = a(\theta)d\phi + b(\theta)d\psi$  winding around the  $\phi\psi$ -torus of a field described by Euler angles  $(\theta, \phi, \psi)$  and time  $\tau$ . The magnetic field tensor  $\mathbf{F}$  is

$$\mathbf{F} = d\mathbf{A} = \partial_\theta a d\theta \wedge d\phi + \partial_\tau a d\tau \wedge d\phi + \partial_\theta b d\theta \wedge d\psi + \partial_\tau b d\tau \wedge d\psi$$

$$\Rightarrow d(*\mathbf{F}) = \{2\partial_\theta \{\tan(\theta/2)\partial_\theta a\} - \frac{1}{2}\tan(\theta/2)\partial_\tau^2 a\} d\theta \wedge d\varphi^2 \wedge d\tau + \{2\partial_\theta \{\cot(\theta/2)\partial_\theta b\} - \frac{1}{2}\cot(\theta/2)\partial_\tau^2 b\} d\theta \wedge d\varphi^1 \wedge d\tau = 0.$$

For  $d\tau \equiv 0$  we recover the time independent solution  $a(\theta) = A \ln|\sin(\theta/2)|$  and  $b(\theta) = B \ln|\sec(\theta/2)|$ . The helicity of the configuration is given by

$$K = \int \mathbf{A} \wedge d\mathbf{A} = 2AB\pi^2 \int_0^\pi \ln|\sec(\theta/2)| \cot(\theta/2) - \ln|\sin(\theta/2)| \tan(\theta/2) d\theta = \frac{AB\pi^4}{3}$$



Plots of the magnetic flux for the time independent and time dependent solutions in 2-D and 3-D. The  $1^{st}$  is a cross-sectional slice along the  $xy$ -plane and the  $3^{rd}$  along the  $xz$ -plane. The time independent solution is geometrically static. We only show one component of the time dependent solution which includes the summation of many fluctuating terms of a Jacobi polynomial. It is in geometric turmoil, however over time both solutions expand to infinity.

For the time dependent solution with  $P_n^{(\alpha,\beta)}(\cos\theta)$  denoting the Jacobi polynomial

$$a(\theta, \tau) = \cos^2(\theta/2) \sum_{n=0}^{\infty} P_n^{(0,1)}(\cos\theta) \{A_n e^{2(n+1)\tau} + B_n e^{-2(n+1)\tau}\}, b(\theta, \tau) = \sin^2(\theta/2) \sum_{m=0}^{\infty} P_m^{(1,0)}(\cos\theta) \{C_m e^{2(m+1)\tau} + D_m e^{-2(m+1)\tau}\}$$

This gives rise to a wave equation for helicity, involving continuous deformations and topological reconnections of flux. Helicity is not conserved and not quantised, unlike the vortex and monopole. The helicity is part of a 2-gerbe, reducing the global union of open covers to the single point origin of a sphere, which is undefined for angles, permitting helicity to take any real value.

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N J Hitchin, Lectures On Special Lagrangian Submanifolds (1999), P A M Dirac, Quantised Singularities in the Electromagnetic Field (1931), P M Bellan, Spheromaks: (2000), D Lovelock & H Rund Tensors, Differential Forms, and Variational Principles(1989), M Nakahara, Geometry, Topology and Physics, (1990), I G Moss, Topology and Gerbes(2007)